Computational Material Forces in Micromorphic Continua

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The micromorphic continuum theory is used to describe materials with significant microstructure which thus exhibit scale-dependence (see e.g. [1], [2] [3]). Microcontinua are assumed to be attached to each physical point and may experience both stretch and rotation which are affine throughout the microcontinuum, nevertheless kinematically independent from the deformation on the macroscale. The additional kinematical quantities which account for the micro-deformation yield additional stresses and contributions to the balance of momentum. Additionally to the common finite-element approximation which here is a coupled problem to be solved for macro- and the micro-quantities, we apply the method of material forces, cf. [4], [5].

1 Introduction

Contrary to the well-known spatial-motion problem, in the material-motion problem the quantities are parametrised with respect to spatial coordinates, \( x \), and the mappings are performed from the spatial to the material configuration. In contrast to the intuitively used spatial force, a material force is an energetically conjugate quantity to a variation in the material position, \( \delta X \), of a particular spatial point \( x \).

2 Kinematics, Balance Relations, Constitutive Assumption

In the micromorphic continuum, we distinguish between the macro- and the microscale. While the macroscale description follows that of the classical continuum, additionally for the microcontinuum a micro-deformation map and its gradient with respect to macro-position are introduced for the material-motion problem, see Figure 1. In the sequel we shall focus on the material-motion problem.

![Figure 1](image)

Fig. 1 Micromorphic deformation maps with respect to the spatial- and the material-motion problem (red and blue, resp.)

Herein the balance of the total energy must account for a release of potential energy upon a variation of the material geometry which necessitates the definition of non-zero material boundary tractions. The stored-energy densities of both configurations are related by \( U_0 = U_1 \det F \), which may directly be transferred to the internally-stored-energy density, since \( U_1 \equiv W_t \) if we neglect body forces. Upon assumption of a hyperelastical material, we may define the Piola-type macro-, micro- and double-stress, \( p := d_f U_t, \bar{p} := d_f U_0, \bar{q} := d_g U_0 \) as being energetically conjugate to the deformation measures defined above. For the material-motion problem the strong form of the balance of momentum is constituted by a macro- and a micro-term and supplemented by the resulting non-zero Neumann boundary conditions:

\[
\begin{align*}
\text{div } p &= 0, \\
\text{div } \bar{q} - \bar{p} &= 0, \\
p \cdot n &=: T^p_t, \\
\bar{q} \cdot n &=: T^\theta_t. 
\end{align*}
\]

(1)

A hyperelastic constitutive formulation incorporation scale-dependence, \( l \), and a scale-transition term, \( p \), is introduced as

\[
W_t = \frac{1}{2} \det f \left[ \lambda \ln^2(\det F) + \mu [F : F - n^{\text{dim}} - 2 \ln(\det F)] + \mu \ell^2 \bar{G} \cdot \bar{G} + p [\bar{F} - F] : [\bar{F} - F] \right].
\]

(2)

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3 Finite-Element Approximation and Material Forces

The macro-map \( \varphi \) and the micro-map \( \bar{F} \) are chosen to be the unknown quantities for which the FE solution is obtained. In a post-processing step, the discrete material-force vector \( \bar{S}_I \) and the discrete configurational-double-force tensor \( \bar{M}_{IJ} \) may be derived utilising the material-motion balance of macro- and micro-momentum (1), respectively:

\[ \bar{S}_I := \int_{g_I} \left[ \bar{q} \cdot \nabla x N^\varphi_I + \bar{p} N^\varphi_I \right] \, \mathrm{d}v \]
\[ \bar{M}_{IJ} := \int_{g_I} \left[ \bar{q} \cdot \nabla x N^f_J + \bar{p} N^f_J \right] \, \mathrm{d}v. \]  (3)

The first is energetically conjugate to variations in the material placement, \( \delta \varphi_I \), while the latter is conjugate to variations in the micro-deformation map, \( \delta \bar{f}_{IJK} \) at the nodes I and J respectively. Isoparametric quadrilateral elements with quadratic shape functions \( N^\varphi_I \) and linear shape functions \( N^f_J \) have been implemented.

4 Numerical Examples

The successful finite-element implementation and the application of the material-force method is displayed here for two different boundary-value problems: first a specimen with a centred circular hole and secondly a specimen with static centred crack, both under uniaxial tension, see Fig 2.

For a significant internal length the deformation around the inhomogeneity is less distinct and thus the material forces are more evenly distributed. Particularly the material force at the crack tip for the latter specimen is smaller than for the classical case (\( l = 0 \)).

5 Conclusion

For the micromorphic continuum without restriction to infinitesimal deformations a continuum-mechanics framework has been briefly presented for which we have successfully derived a finite-element approximation and the material forces and configurational double forces. The latter require further investigations from which we hope to obtain more insight into the propagation of discontinuities in a micromorphic continuum.

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References