A view on anisotropic finite hyper-elasticity

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Abstract

The paper presents a modular framework for the formulation of anisotropic hyper-elastic materials at large strains. Thereby, additional symmetric second order tensors are incorporated into the free Helmholtz energy density which allow, e.g., the interpretation as structural tensors. In order to prove the analogy between the material setting – usually based on the right Cauchy–Green tensor – and the spatial formulation – typically in terms of the Finger tensor – the general representation theorem of isotropic tensor functions is applied. As a result, the spatial formats of the stress tensors and the tangent operators within the anisotropic hyper-elastic case are given, whereby a specific additive structure of contributions due to the Finger tensor and the additional symmetric tensors is obtained.

Keywords: Anisotropy; Structural tensors; Large strains

1. Introduction

The description and modelling of anisotropic material behaviour is of cardinal importance in material and engineering science. Various different approaches to formulate appropriate constitutive equations have been discussed in the literature. Besides micro-mechanically motivated procedures, several strategies to incorporate anisotropy into the standard framework of non-linear continuum mechanics were established. In this work, the modification of the free Helmholtz energy density compared to the isotropic case is considered, with special emphasis on an Eulerian setting whereby we restrict ourselves to the isothermal and non-dissipative case for clarity’s sake.

The formulation of anisotropic materials at large strains refers usually to a material framework, e.g., in terms of the right Cauchy–Green tensor or is alternatively related to the spatial metric tensor (as a consequence of the covariance of the free Helmholtz energy density). One frequently applied approach to model anisotropy is based on the incorporation of additional tensorial arguments into the free Helmholtz energy density – typically structural tensors, for an overview see, e.g., the contributions in Boehler (1987). On this basis, the main goal of this contribution is to give an anisotropic framework in terms of the Finger tensor at hand which allows convenient application within a computational setting, e.g., a finite element formulation. We obtain in particular an anisotropic version of the established Murnaghan–Truesdell formula as highlighted in Murnaghan (1937) and Truesdell and Noll (1965, Eq. (85.15)). Based on the introduction of an additional tensor series of symmetric second order tensors into the free Helmholtz energy density, a typical additive decomposition into the standard “isotropic” formula and additional “anisotropic” contributions is observed. Thereby, the representation theorem of isotropic tensor functions serves as the key ingredient for the proof of the sought stress relations and elastic tangent operators. Alternatively, the fundamental covariance relation can be applied, compare Lu and Papadopoulos (2000).

The paper is organised as follows: to set the stage, we summarise basic essentials of non-linear continuum mechanics in Section 2 and introduce some of the notations used in the sequel. Further comments on the notation are given in Appendix A. Section 3 reiterates an established representation of anisotropic hyper-elasticity in terms of the spatial co-variant metric tensor or the right Cauchy–Green tensor, respectively. Subsequently, the concept of the anisotropic approach is highlighted in Section 4. As the main contribution of this work, we deduce representations of the spatial and material stress tensors and tangent operators.
Thereby, as one possibility of the proof, the representation theorem of isotropic tensor functions is the essential point of departure. The alternative proof is reiterated in Appendix C. Finally, a numerical example within a finite element setting highlights the applicability of the proposed framework, Section 5.

1.1. Conventions and notation

In this contribution, a tensor notation in the spirit of Marsden and Hughes (1983) is applied since we adopt the point of view that “Even for a simple body, tensor analysis on manifolds clarifies the basic theory. For instance, using manifold ideas we can see clearly how to formulate the pull-back and push-forward . . .” (Marsden and Hughes, 1983, Chapter 1, Box 2.1). In addition to standard conventions, we introduce the symbol \( \natural \) to indicate mixed-variant tensors. Apparently, the spatial identity (with the Kronecker delta \( \delta_{ij} \) in Cartesian coordinates) will read \( g^i_i = 1 \) in the sequel. The scalar product and the tensor product of vectors – say \( v^i_1, v^i_2, v^i_3 \) – are written in standard fashion, namely \( v^i_1 \cdot v^i_2 = v^2_1 \cdot v^2_2 \) and \( [v^i_1 \otimes v^i_2] \cdot [v^i_3 \otimes v^i_4] = [v^2_1 \cdot v^2_2] [v^3_1 \cdot v^3_2] \). Thereby each \( \cdot \) indicates one contraction. The trace operation for second order tensors is consequently determined by, e.g., \( \text{tr}(g^i_i \{ b^i_i \}^3) = g^i_i \cdot b^i_i \) whereby the notation \( \{ b^i_i \}^3 \) characterises the transposed field.

To give an example, we highlight the previously mentioned relation for the Kirchhoff stress tensor within an isotropic setting which allows the following representations

\[
\begin{align*}
\text{Cartesian coordinates} & \quad \tau_{ij} = 2 \partial_{b_{ik}} \psi_0 \partial_{b_{kj}}, \\
\text{general coordinates} & \quad \tau^{ij} = 2 g^{ik} \partial_{b_{kl}} \psi_0 b_{lj}, \\
\text{general tensor notation} & \quad \tau^i = 2 g^i \cdot \partial_{b_{kj}} \psi_0 \cdot b^j
\end{align*}
\]

whereby the free Helmholtz energy density \( \psi_0 \) represents an isotropic scalar-valued tensor function within this contribution, see, e.g., Šilhavý (1997, Section 8.2). Thus, assuming the set of arguments abbreviated by \( [\bullet] \), the fundamental relation

\[
\psi_0([\bullet]; X) = \psi_0(Q^2 \cdot [\bullet]; X)
\]

holds, with \( Q^2 \in \mathbb{O}_3 \) characterising an orthogonal tensor of second order and moreover, the notation \( \bullet \) indicates the appropriate linear transformation.

2. Kinematics

Let the reference configuration of the body of interest be denoted by \( B_0 \subset \mathbb{E}^3 \) and the corresponding spatial complement by \( B_t \subset \mathbb{E}^3 \). The non-linear deformation map \( \varphi(X, t): B_0 \times \mathbb{R} \to B_t \) maps material points \( X \in B_0 \) onto spatial points \( x = \varphi(X, t) \in B_t \). By introducing these points \( x = x(\theta^i) \) and \( X = X(\theta^i) \) in terms of convective, curvilinear coordinates \( \theta^i = \theta^i(x, t) \) and \( \theta^i = \Theta^i(X) \) one obtains the following line elements

\[
\begin{align*}
\varphi_i &= \partial_{x^i} x \circ T^* B_t \to \mathbb{R}, \\
g_i &= \partial_{\theta^i} \varphi \circ T^* B_t \to \mathbb{R}, \\
G_i &= \partial_X \Theta^i \circ T^* B_0 \to \mathbb{R}, \quad G^i = \partial_V \Theta^i \circ T^* B_0 \to \mathbb{R}
\end{align*}
\]

which are identified as natural and dual base vectors and span the tangent and co-tangent (dual) spaces, respectively. Thus, the spatial and material metric tensors follow straightforward as

\[
\begin{align*}
g^i_i &= \delta_{ij} \circ T^* B_t \times T^* B_t \to \mathbb{R}, \\
g^i_j &= g_{ij} \cdot g_{j}, \\
G^i_i &= G_{ij} \cdot G_{j}, \\
G^i_j &= G_{ij} \cdot G_{j}
\end{align*}
\]

(2)

In addition, we apply mixed-variant spatial and material identity-tensors in terms of the natural and dual base vectors, i.e.,

\[
\begin{align*}
\varphi_i &= \varphi_i \circ T^* B_t \times T^* B_t \to \mathbb{R}, \\
G^i &= G_i \circ G^i \circ T^* B_0 \times T^* B_0 \to \mathbb{R}, \quad G^i_i = G_{ij} \cdot G_{j}
\end{align*}
\]

for more background informations we refer the reader to the work by Lodge (1974, Chapters 2 and 11). The linear tangent map with respect to \( \varphi \) is consequently a mixed-variant two-point tensor (deformation gradient)

\[
\begin{align*}
\mathbf{F}^i &= \partial_X \varphi = \partial_{y^i} \varphi \otimes \partial_x \Theta^i \circ \Phi = \varphi_{i} \circ \varphi^i \circ T^* B_0 \to T^* B_t
\end{align*}
\]

(4)

which transforms tangent vectors to material curves into tangent vectors to spatial curves. Thereby, for sake of notational simplicity, the inverse non-linear deformation map \( \Phi(x, t) = \varphi^{-1}_i: B_t \times \mathbb{R} \to B_0 \) has been introduced. Apparently, we claim
The dualism highlighted in Eq. (7) represents the fundamental covariance relation of the free Helmholtz energy functional. Let \( \Phi \) denote an arbitrary spatial and material diffeomorphism with \( \Phi \neq \Psi^{-1} \) and \( \Omega \equiv \omega^{-1} \), respectively. Then “spatial” covariance reads as

\[
\psi_{0}(g^b, b^g, \chi_{\Omega}; \chi) = \psi_{0}(\chi^b, \psi \circ F \circ \chi, G^b, \chi_{\Omega}; \chi).
\]
The specific choice $\Psi(x, t) = \Phi(x, t)$ results in the conclusion that $F^\flat$ enters $\psi_0$ via $C^\flat$ (note that $\Phi \ast F^\flat = G^\flat$ is redundant),

$$\psi_0(g^\flat, F^\flat, G^\flat, A^\flat_{1,...,n}; X) = \psi_0(C^\flat, G^\flat, A^\flat_{1,...,n}; X).$$

(10)

Furthermore, Eq. (9) boils down to the axiom of material frame indifference (as represented by invariance with respect to superposed rigid body motions, Hooke–Poisson–Cauchy form) if $\Psi(x, t)$ denotes a regular orientation preserving spatial isometry

$$\psi_0(g^\flat, F^\flat, G^\flat, A^\flat_{1,...,n}; X) = \psi_0(q^\flat \ast g^\flat, q^\flat \ast F^\flat, G^\flat, A^\flat_{1,...,n}; X) \quad \forall q^\flat \in \mathbb{O}^3_+$$

(11)

with $q^\flat \ast g^\flat = g^\flat$ being obvious, compare, e.g., Marsden and Hughes (1983, Chapter 3.2, Axiom 2.9) Next, superposing a material diffeomorphism $\omega(X, t)$ onto $\psi_0^g$ as highlighted in Eq. (10) we obtain

$$\psi_0^g(C^\flat, G^\flat, A^\flat_{1,...,n}; X) = \psi_0^g(\Omega^\ast C^\flat, \omega_s G^\flat, \omega_s A^\flat_{1,...,n}; X).$$

(12)

Similar to Eq. (10), the embodied option $\omega(X, t) = \varphi(X, t)$ results in Eq. (7),

$$\psi_0^g(\Phi^\ast C^\flat, \varphi_s G^\flat, \varphi_s A^\flat_{1,...,n}; X) = \psi_0^g(g^\flat, b^\flat, a^\flat_{1,...,n}; X).$$

(13)

and finally, by choosing $\omega(X, t)$ to represent a material isometry, we end up with the definition of a scalar-valued isotropic tensor function

$$\psi_0^g(C^\flat, G^\flat, A^\flat_{1,...,n}; X) = \psi_0^g(Q^\flat \ast C^\flat, Q^\flat \ast G^\flat, Q^\flat \ast A^\flat_{1,...,n}; X) \quad \forall Q \in \mathbb{O}^3$$

(14)

with $Q^\flat \ast G^\flat = G^\flat$ being obvious.

3.1. General spatial and material stress relation

Within the standard argumentation of rational thermodynamics the free Helmholtz energy density defines the stress tensors $\tau^\flat$ (Kirchhoff stress) and $S^\flat$ (second Piola–Kirchhoff stress) as

$$\tau^\flat = 2\partial_{g^\flat} \psi_0^g(g^\flat, b^\flat, a^\flat_{1,...,n}; X): T^\ast B_0 \times T^\ast B_0 \to \mathbb{R},$$

$$S^\flat = 2\partial_{C^\flat} \psi_0^g(C^\flat, G^\flat, A^\flat_{1,...,n}; X): T^\ast B_0 \times T^\ast B_0 \to \mathbb{R}.$$  

(15)

3.1.1. Derivation of the general spatial stress relation

The isothermal Clausius–Duhem inequality in local form for a purely elastic process reduces to an equality which results within the spatial setting in

$$D_\tau^\flat = [m^\flat]^d : I^\flat = D_\tau \psi_0^g(g^\flat, b^\flat, a^\flat_{1,...,n}; X) = 0$$

(16)

whereby $[m^\flat]^d = g^\flat \ast \tau^\flat : T^\ast B_0 \times T^\ast B_0 \to \mathbb{R}$ denotes a mixed-variant spatial stress tensor of Mandel-type, $D_\tau \psi_0^g$ represents the material time derivative and $I^\flat = D_t F^\natural \cdot F^\natural : T^\ast B_0 \times T^\ast B_0 \to \mathbb{R}$ defines the spatial gradient of the physical velocity, see, e.g., Truesdell and Noll (1965, Chapter D II a). The stress power $\mathcal{W}_0^\flat$ can alternatively be written in the format

$$\mathcal{W}_0^\flat = [m^\flat]^d : I^\flat = [m^\flat \cdot g^\flat]^d : [g^\flat \cdot I^\flat] = \tau^\flat \cdot [g^\flat \cdot I^\flat]_{\text{sym}} = \frac{1}{2} \tau^\flat \cdot L_g g^\flat$$

(17)

with the notation $L_g \psi_0^g = \partial_{g^\flat} \psi_0^g : L_g g^\flat = 2\partial_{g^\flat} \psi_0^g : [g^\flat \cdot I^\flat]_{\text{sym}} = 2[g^\flat : \partial_{g^\flat} \psi_0^g] : I^\flat$.

(18)

whereby the application of the Lie-derivative $L_g g^\flat$ stems from the push-forward of the material time derivative $D_t C^\flat$. Obviously Eqs. (16) and (18) define the mixed-variant stress tensor as

$$[m^\flat]^d = 2 g^\flat \cdot \partial_{g^\flat} \psi_0^g,$$

(19)

which results, in view of Eq. (17), in the well-known Doyle–Ericksen formula $\tau^\flat = 2\partial_{g^\flat} \psi_0^g$. 


3.1.2. Derivation of the general material stress relation

The material format of Eq. (19) can be derived by standard push-forward and pull-back operations or via

\[ D^0_\alpha = [M^2]^d : L^2 - D_t \psi^0_0(\ell^0, G^2, A^2_1,\ldots,n; X) = 0 \] (20)

whereby \([M^2]^d = C^\circ \cdot S^2 : T B_0 \times T^* B_0 \rightarrow R\) denotes the Mandel stress tensor and \(L^2 = f^2 \cdot D_t F^2 : T^* B_0 \times T B_0 \rightarrow R\) represents the mixed-variant pull-back of the physical velocity gradient. Now, taking the relation

\[ \forall \psi_0^0 = [M^2]^d : L^2 = [M^2 \cdot B^2]^1 : [C^\circ \cdot L^2] = S^2 : [C^\circ \cdot L^2]_{sym} = \frac{1}{2} S^2 : D_t C^\circ \]

and \(D_t \psi^0_0 = \partial_{C^\circ} \psi^0_0 \cdot D_t C^\circ\) into account, the material hyper-elastic law reads

\[ [M^2]^d = 2C^\circ : \partial_{C^\circ} \psi^0_0 \] (22)

incorporating the usual format of the second Piola–Kirchhoff stress tensor, \(S^2 = 2\partial_{C^\circ} \psi^0_0\).

3.2. General spatial and material tangent operator

For an isothermal hyper-elastic setting it is straightforward to show that the spatial and material tangent operators \(\ell^2\) and \(E^2\) are Hessians of the free Helmholtz energy density. Due to the non-dissipative deformation induced character, appropriate time derivatives of the elements of the additional tensor series must vanish – \(L_t A^2_1,\ldots,n = 0\) and \(D_t A^2_1,\ldots,n = 0\) – and the typical format of the Hessians reads

\[ \ell^2 = 4a^2_{\ell^0 \otimes e^0} \psi^0_0(\ell^0, b^2, a^2_1,\ldots,n; X) : T^* B_0 \times T B_0 \times T^* B_0 \times T B_0 \rightarrow R. \]

\[ E^2 = 4a^2_{\ell^0 \otimes e^0} \psi^0_0(\ell^0, G^2, A^2_1,\ldots,n; X) : T^* B_0 \times T B_0 \times T^* B_0 \times T B_0 \rightarrow R. \] (23)

3.2.1. Derivation of the general spatial and material tangent operator

For completeness, we give an outline of the derivation of the tangent operators which appear in the relations \(L_t \tau^2 = \frac{1}{2} \ell^2 : \ell^0 g^0\) and \(D_t S^2 = \frac{1}{2} E^2 : D_t C^\circ\). The material version yields

\[ D_t S^2 = \partial_{C^\circ} [2\partial_{C^\circ} \psi^0_0] : D_t C^\circ = 2\partial^2_{C^\circ \otimes C^\circ} \psi^0_0 : D_t C^\circ \]

which proves Eq. (23)1. Now, standard operations as \(D_t S^2 = \Phi_\circ L_t \tau^2\), \(D_t C^\circ = \psi^0_0 L^0 g^0\) and

\[ \partial_{C^\circ} \psi^0_0 = \partial_{C^\circ} \psi^0_0 \iff \partial_{C^\circ} \ell^2 = f^2 \cdot \partial_{C^\circ} \ell^0_0 : [f^2]^d, \]

\[ \partial^2_{C^\circ \otimes C^\circ} \psi^0_0 = \partial_{C^\circ} f^2 \cdot \partial_{C^\circ} \psi^0_0 : [f^2]^d \cdot \partial_{C^\circ} g^0 = [f^2 \otimes f^2] : \partial_{C^\circ} \psi^0_0 : [f^2]^d \]

finally yield the spatial tangent operator in Eq. (23)1, see Appendix A for notational details.

4. Representation of anisotropy

As previously mentioned, anisotropy possibly enters the formulation if at least one non-spherical tensor \(A^2_1\) is employed to the free Helmholtz energy density. Eqs. (15) and (23) define the stress tensors and tangent operators without loss of generality, but in addition we seek an analogous expression in terms of derivatives with respect to \(b^2, a^2_1,\ldots,n\) or \(G^2, A^2_1,\ldots,n\).

4.1. Anisotropic spatial and material stress relation

As one emphasis of this contribution, we make the

**Proposition 4.1.** For an anisotropic setting based on \(\psi^0_0(\ell^0, b^2, a^2_1,\ldots,n; X) = \psi^0_0(\ell^0, G^2, A^2_1,\ldots,n; X)\), the stress tensor (as represented by the Kirchhoff or second Piola–Kirchhoff stress) can alternatively be formulated in terms of \(b^2, a^2_1,\ldots,n\) or \(G^2, A^2_1,\ldots,n\). The formal structure is similar to the established Murnaghan–Truesdell formula for an
 isotropic constitutive equation since the anisotropic formulation results in a specific additive decomposition into derivatives in terms of  and  respectively. Anticipating the result, we will end up with

\[
\tau^0 = 2g^0 \cdot \nabla_p \psi_0^0 - b^0 + 2 \sum_{p=1}^{n} \nabla_p \cdot \nabla_p^0 \cdot a_p^0 = [\tau^0]^1.
\]

\[
S^0 = 2B^0 \cdot \nabla_p \psi_0^0 \cdot G^0 + 2 \sum_{p=1}^{n} B^0 \cdot \nabla_p \psi_0^0 \cdot A_p^0 = [S^0]^1.
\]

The corresponding proof can be based on the general representation theorem of scalar-valued isotropic tensor functions. Thereby, one has to express every obtained stress generator based on  or  in terms of sums of generators based on  and  or  respectively.

4.1.1. Derivation of the anisotropic spatial stress relation

The local form of the isothermal Clausius–Duhem inequality within the anisotropic spatial setting for a non-dissipative process (,  and  with  and  specifies the terminology material or rather deformation induced anisotropy.

\[
D_p^i = [m^d] : l^i - D_i \psi_0^0 (g^0, b^0, a^0_{1,\ldots,n}; X) = [m^d] : l^i - \partial_{b^0} \psi_0^0 : D_i b^0 - \sum_{p=1}^{n} \partial_{a_p^0} \psi_0^0 : D_i a_p^0
\]

\[
= [m^d] : l^i - 2\partial_{b^0} \psi_0^0 : [l^i \cdot b^0]^\text{sym} - 2 \sum_{p=1}^{n} \partial_{a_p^0} \psi_0^0 : [l^i \cdot a_p^0]^\text{sym} = 0.
\]

whereby symmetry relations have been taken into account. Thus, following the standard argumentation of rational thermodynamics, a definition of the mixed-variant Mandel-type stress tensor can be constructed as

\[
[m^d] = 2\partial_{b^0} \psi_0^0 \cdot b^0 + 2 \sum_{p=1}^{n} \partial_{a_p^0} \psi_0^0 \cdot a_p^0.
\]

The fact that the (material) elements of the tensor series  are assumed to stay constant during the deformation process, suggests the terminology material or rather deformation induced anisotropy.

4.1.2. Derivation of the anisotropic material stress relation

The material format of Eq. (28) can be obtained by standard pull-back operations and reads as

\[
[M^d] = 2\partial_{b^0} \psi_0^0 \cdot G^0 + 2 \sum_{p=1}^{n} \partial_{A_p^0} \psi_0^0 \cdot A_p^0.
\]

4.1.3. Derivation of the anisotropic spatial and material stress relation via invariants

In order to prove the symmetry of the Kirchhoff stress tensor  or the second Piola–Kirchhoff stress tensor respectively, within the anisotropic framework highlighted in Eqs. (28) and (29), we take the general representation theorem of scalar-valued isotropic tensor functions into account (an alternative proof is reiterated in Appendix C).

The set of irreducible invariants and the corresponding derivatives (generators) are given in Appendix B. Here, we choose for the denomination of the set of invariants  whereby the notation  is applied. Since the free Helmholtz energy density  represents a scalar-valued isotropic tensor function, we consequently end up with the alternative expressions

\[
\tau^0 = 2 \sum_{k} \partial_{l_k} \psi_0^0 \partial_{g^0} l_k = [\tau^0]^1.
\]

\[
\tau^0 = 2 \sum_{k} \partial_{l_k} \psi_0^0 \partial_{g^0} l_k \cdot b^0 + 2 \sum_{k=1}^{n} \partial_{l_k} \psi_0^0 \partial_{a_p^0} l_k \cdot a_p^0 = [\tau^0]^2.
\]
with respect to the spatial setting, compare Eqs. (15) and (26). In this context, Proposition 4.1 for the anisotropic stress relation (Eq. (26)) is one-to-one with

\[
\begin{align*}
\partial g \cdot I_k &= g^p \cdot \partial g \cdot b^p + \sum_{p=1}^n g^p \cdot \partial a^p \cdot I_k \cdot a^p = b^p \cdot \partial g \cdot g^p + \sum_{p=1}^n a^p \cdot \partial a^p \cdot I_k \cdot g^p, \\
\partial e \cdot I_k &= B^p \cdot \partial g \cdot I_k \cdot B^p + \sum_{p=1}^n B^p \cdot \partial a^p \cdot I_k \cdot A^p = g^p \cdot \partial g \cdot I_k \cdot B^p + \sum_{p=1}^n A^p \cdot \partial a^p \cdot I_k \cdot B^p.
\end{align*}
\]

(31)

In the following, we place emphasis on the spatial setting (the proof for the material framework is of course analogous) and verify Proposition 4.1 in the form of Eq. (31)\textsubscript{1}. After some lengthy but straightforward algebra which is based essentially on Eqs. (B.3)–(B.5), we obtain

(i) \[ \partial g \cdot I_i = g^p \cdot \partial g \cdot I_i \cdot b^p + \sum_{p=1}^n g^p \cdot \partial a^p \cdot I_i \cdot a^p = g^p \cdot g^p \cdot b^p + 0 = b^p, \]

(ii) \[ \partial g \cdot I_{ii} = g^p \cdot \partial g \cdot I_{ii} \cdot b^p + \sum_{p=1}^n g^p \cdot \partial a^p \cdot I_{ii} \cdot a^p = 2g^p \cdot g^p \cdot b^p \cdot g^p \cdot b^p + 0 = 2b^p \cdot g^p \cdot b^p, \]

(iii) \[ \partial g \cdot I_{iii} = g^p \cdot \partial g \cdot I_{iii} \cdot b^p + \sum_{p=1}^n g^p \cdot \partial a^p \cdot I_{iii} \cdot a^p = 3g^p \cdot g^p \cdot b^p, \]

(iv) \[ \partial g \cdot I_{iv} = g^p \cdot \partial g \cdot I_{iv} \cdot b^p + \sum_{p=1}^n g^p \cdot \partial a^p \cdot I_{iv} \cdot a^p = 0 + g^p \cdot g^p \cdot g^p \cdot b^p = a^p, \]

(v) \[ \partial g \cdot I_{v} = g^p \cdot \partial g \cdot I_{v} \cdot b^p + \sum_{p=1}^n g^p \cdot \partial a^p \cdot I_{v} \cdot a^p = g^p \cdot g^p \cdot b^p \cdot b^p \cdot g^p \cdot b^p + g^p \cdot g^p \cdot b^p \cdot a^p \cdot a^p = [b^p \cdot g^p \cdot b^p]_{\text{sym}}, \]

(vi) \[ \partial g \cdot I_{vi} = g^p \cdot \partial g \cdot I_{vi} \cdot b^p + \sum_{p=1}^n g^p \cdot \partial a^p \cdot I_{vi} \cdot a^p = 2g^p \cdot g^p \cdot g^p \cdot b^p \cdot a^p \cdot a^p = 2b^p \cdot g^p \cdot a^p \cdot a^p \]

which proves Proposition 4.1. The outline of the transposed version – Eq. (31)\textsubscript{2} – being obvious (note that the sum over all contributions in Eq. (30)\textsubscript{2} results in a symmetric Kirchhoff stress while single contributions are non-symmetric in general).

**Remark 4.1.** Apparently, an isotropic setting is obtained if the additional tensor series are neglected, \( A_i^1, ..., n = 0_{i, j, k} \), which results in

\[
\begin{align*}
\tau^i &= 2g^p \cdot \partial g \cdot \psi^i_0 (g^p, b^p; X) \cdot b^p = [\tau^i]_{\text{sym}}, \\
S^i &= 2B^p \cdot \partial g \cdot \psi^i_0 (C^p, G^p; X) \cdot G^p = [S^i]_{\text{sym}},
\end{align*}
\]

(32)

whereby the spatial format is well-established in the literature, see, e.g., Truesdell and Noll (1965, Eq. (85.15)). Furthermore, note that the specific choice of spherical tensors, \( A_i^1, ..., n \propto G^p \), or the incorporation of scalar-valued fields instead of second order tensors, results similarly in an isotropic constitutive equation.

4.2. Anisotropic spatial and material tangent operator

On the basis for the highlighted stress relation, the spatial and material tangent operator within a non-dissipative anisotropic setting can be derived in terms of \( b^p \) and \( a^p \) or \( G^p \) and \( A^p \), respectively. Anticipating the result, again a specific additive structure is obtained, i.e.,

\[
\begin{align*}
e^p &= 4[b^p \otimes g^p] \cdot [\partial g \otimes \psi^i_0] + 4 \sum_{p=1}^n [a^p \otimes g^p] \cdot [\partial a^p \otimes \psi^i_0] + 4 \sum_{p=1}^n [a^p \otimes g^p] \cdot [\partial a^p \otimes \psi^i_0] + 4 \sum_{p=1}^n [a^p \otimes g^p] \cdot [\partial a^p \otimes \psi^i_0],
\end{align*}
\]

(32)
\[ E^x = 4 \left[ G^x \otimes B^x \right] : \partial^2_{G^x \otimes G^x} \psi^0_0 \left[ B^x \otimes G^x \right] + 4 \sum_{p=1}^{n} \left[ A^p_{B^x} \otimes B^x \right] : \partial^2_{A^p_{G^x} \otimes G^x} \psi^0_0 \left[ B^x \otimes G^x \right] \\
+ 4 \sum_{p=1}^{n} \left[ G^x \otimes B^x \right] : \partial^2_{G^x \otimes A^p_{B^x}} \psi^0_0 \left[ B^x \otimes A^p_{B^x} \right] + 4 \sum_{q,p=1}^{n} \left[ A^p_{B^x} \otimes B^x \right] : \partial^2_{A^p_{A^p_{B^x}} \otimes A^p_{B^x}} \psi^0_0 \left[ B^x \otimes A^p_{B^x} \right]. \]  

(33)

4.2.1. Derivation of the anisotropic spatial tangent operator

The derivation of the spatial tangent operator within a non-dissipative \((L \sim 0, n \sim 1, \ldots, n)\) and anisotropic setting is based essentially on Eq. (26). In addition, the relations \(\tau^x = g^x \cdot [m^x]^d\) and \(D_t b^x = 2l^x \cdot a^x_{sym}\), \(D_t a^x_{sym} = 2l^x \cdot a^x_{sym}\) are taken into account. In this context, the material time derivative of the Kirchhoff stress tensor

\[ D_t \tau^x = 2a^x_{sym} \cdot \tau^x : [l^x \cdot b^x] + 2 \sum_{p=1}^{n} \partial_p \cdot \tau^x : [l^x \cdot a^x_{sym}] \]  

(34)

takes the following format by incorporating Eq. (26)

\[ D_t \tau^x = 2a^x_{sym} \cdot \tau^x : [l^x \cdot b^x] + 2 \sum_{p=1}^{n} \partial_p \cdot \tau^x : [l^x \cdot a^x_{sym}] \]

\[ = 2[b^x \otimes g^x] : \partial^2_{b^x \otimes g^x} \psi^0_0 \left[ l^x \cdot b^x \right] \]

\[ + 2 \sum_{q=1}^{n} [b^x \otimes g^x] : \partial^2_{b^x \otimes g^x} \psi^0_0 \left[ l^x \cdot a^x_q \right] \]

\[ + 2 \sum_{q=1}^{n} [g^x \otimes b^x] : \partial^2_{g^x \otimes b^x} \psi^0_0 \left[ g^x \otimes b^x \right] \]

\[ + 2 \sum_{q=1}^{n} [a^x_q \otimes g^x] : \partial^2_{a^x_q \otimes g^x} \psi^0_0 \left[ l^x \cdot a^x_q \right] \]

\[ + 2 \sum_{q=1}^{n} [g^x \otimes b^x] : \partial^2_{g^x \otimes b^x} \psi^0_0 \left[ g^x \otimes b^x \right] + 2[l^x \cdot \tau^x]_{sym}. \]  

(35)

Now, computing the Lie derivative of the Kirchhoff stress tensor, i.e., \(L_r \tau^x = D_t \tau^x - 2[l^x \cdot \tau^x]_{sym}\) (keeping the specific non-dissipative case in mind), in order to construct the spatial tangent operator \(e^x\), we end up with

\[ L_r \tau^x = 2[b^x \otimes g^x] : \partial^2_{b^x \otimes g^x} \psi^0_0 \left[ l^x \cdot g^x \cdot b^x \right] + 2[b^x \cdot [l^x]^d] \cdot g^x \cdot b^x : \partial^2_{b^x \otimes g^x} \psi^0_0 \left[ g^x \otimes b^x \right] \\
+ 2 \sum_{q=1}^{n} [a^x_q \otimes [l^x]^d \cdot g^x \cdot b^x] : \partial^2_{a^x_q \otimes [l^x]^d \cdot g^x \cdot b^x} \psi^0_0 \left[ g^x \otimes b^x \right] + 2 \sum_{q=1}^{n} [b^x \otimes [l^x]^d] \cdot g^x \cdot b^x : \partial^2_{b^x \otimes [l^x]^d \cdot g^x \cdot b^x} \psi^0_0 \left[ g^x \otimes a^x_q \right] \\
+ 2 \sum_{q=1}^{n} [g^x \otimes a^x_q] : \partial^2_{g^x \otimes a^x_q} \psi^0_0 \left[ l^x \cdot a^x_q \right] \]

\[ = 4[b^x \otimes g^x] : \partial^2_{b^x \otimes g^x} \psi^0_0 \left[ g^x \otimes b^x \right] + 4 \sum_{p=1}^{n} [a^x_p \otimes g^x] : \partial^2_{a^x_p \otimes g^x} \psi^0_0 \left[ g^x \otimes b^x \right] \]
Therefore, in view of the spatial setting, the derivatives of the stress generators due to
and can be verified after some lengthy algebra
with Eqs. (31) at hand, the relations
Some straightforward algebra finally yields Eq. (33) 2 .

4.2.3. Derivation of the anisotropic spatial and material tangent operator via invariants

In the sequel, we focus on the spatial setting with respect to a representation in invariants similar to the contributions of the stress tensors in Eq. (31). Obviously, two different families of derivatives are incorporated which can easily be seen in the usual spatial format in terms of the spatial metric tensor
which proves Eq. (33) 1 .

4.2.3. Derivation of the anisotropic spatial and material tangent operator via invariants

In the sequel, we focus on the spatial setting with respect to a representation in invariants similar to the contributions of the stress tensors in Eq. (31). Obviously, two different families of derivatives are incorporated which can easily be seen in the usual spatial format in terms of the spatial metric tensor

\[
4 \partial^{2}_{g} \psi_{0}^{i} = 4 \sum_{k} \partial_{g} \psi_{0}^{i} \partial^{2}_{g} g_{k} \otimes g^{k} \otimes g_{l} = 4 \sum_{k} \partial_{g} \psi_{0}^{i} \partial^{2}_{g} g_{k} \otimes g^{k} \otimes g_{l}
\]

with \( k, l = (i), \ldots, (vi) \), compare Appendix B. The second type of contributions in Eq. (38) incorporates dyadic products of the stress generators. Thus, with Eqs. (31) at hand, the relations

\[
\partial_{g} \psi_{0}^{i} \partial_{g} \psi_{0}^{j} = \left[ g^{k} \cdot \partial_{g} \psi_{0}^{i} \right] \left[ g^{l} \cdot \partial_{g} \psi_{0}^{j} \right] + \left[ g^{k} \cdot \partial_{g} \psi_{0}^{j} \right] \left[ g^{l} \cdot \partial_{g} \psi_{0}^{i} \right]
\]

hold. Next, the corresponding relation due to the first contribution in Eq. (38) is verified via the representation theorem. Therefore, in view of the spatial setting, the derivatives of the stress generators due to \( g^{k}, b^{k} \) and \( a^{i} \) have to be computed which are given in Appendix B, Eqs. (B.6)–(B.8). Now, the relation for the second derivatives of the invariants reads as

\[
\partial^{2}_{g} \psi_{0}^{i} \partial_{g} \psi_{0}^{j} = \left[ g^{k} \cdot \partial_{g} \psi_{0}^{i} \right] \left[ g^{l} \cdot \partial_{g} \psi_{0}^{j} \right] + \sum_{q=1}^{n} \left[ a_{p}^{q} \otimes g^{k} \right] \left[ g^{l} \cdot \partial_{g} \psi_{0}^{i} \right] \left[ g^{k} \cdot \partial_{g} \psi_{0}^{j} \right]
\]

and can be verified after some lengthy algebra.
\[ \begin{align*}
\text{(ii)} & \quad \partial^2_{g^2 \otimes b^2} I_{(ii)} = \partial^2_{b^2 \otimes b^2} I_{(ii)} + \sum_{q=1}^{n} \partial^2_{a^2_q \otimes a^2_q} I_{(ii)} = [g^2 \otimes b^2]^{\otimes} + \sum_{p=1}^{n} [a^2_p \otimes b^2]^{\otimes} + \sum_{p,q=1}^{n} [a^2_p \otimes a^2_q]^{\otimes} \\
& \quad = [g^2 \otimes b^2]^{\otimes} + [g^2 \otimes b^2]^{\otimes} + 0 + 0 + 0 = [b^2 \otimes b^2]^{\otimes}.
\end{align*} \]

\[ \begin{align*}
\text{(iii)} & \quad \partial^2_{g^2 \otimes g^2} I_{(iii)} = [b^2 \otimes b^2]^{\otimes} + \sum_{q=1}^{n} [b^2 \otimes a^2_q]^{\otimes} + [b^2 \otimes b^2]^{\otimes} + \sum_{p,q=1}^{n} [a^2_p \otimes a^2_q]^{\otimes} \\
& \quad = 3[b^2 \otimes b^2]^{\otimes} + [b^2 \otimes b^2]^{\otimes} + [b^2 \otimes g^2]^{\otimes} + [g^2 \otimes b^2]^{\otimes} + [g^2 \otimes a^2_q]^{\otimes} \\
& \quad = 0 + 0 + 0 + 0 + 0 = 0.
\end{align*} \]

\[ \begin{align*}
\text{(v)} & \quad \partial^2_{g^2 \otimes g^2} I_{(v)} = [b^2 \otimes g^2]^{\otimes} + \sum_{q=1}^{n} [b^2 \otimes a^2_q]^{\otimes} + \sum_{p,q=1}^{n} [a^2_p \otimes a^2_q]^{\otimes} \\
& \quad = 0 + 0 + 0 = 0.
\end{align*} \]

\[ \begin{align*}
\text{(vi)} & \quad \partial^2_{g^2 \otimes g^2} I_{(vi)} = [b^2 \otimes g^2]^{\otimes} + \sum_{q=1}^{n} [b^2 \otimes a^2_q]^{\otimes} + \sum_{p,q=1}^{n} [a^2_p \otimes a^2_q]^{\otimes} \\
& \quad = 0 + 0 + 0 = 0.
\end{align*} \]

which proves essentially the spatial setting of Eq. (33). The outline of the material version is straightforward and thus not highlighted here.
Remark 4.2. In analogy to Remark 4.1 the specific isotropic case with \( A^\sharp_{1,...,n} = 0_{1,...,n} \) yields
\[
\begin{align*}
e^\sharp & = 4\left[ G^\sharp \otimes G^\sharp \right] : \partial^2_{\psi}\circ\psi_0(g^\sharp, b^\sharp; X) : \left[ g^\sharp \otimes b^\sharp \right], \\
E^\sharp & = 4\left[ G^\sharp \otimes B^\sharp \right] : \partial^2_{\psi}\circ\psi_0(C^\sharp, G^\sharp; X) : \left[ B^\sharp \otimes G^\sharp \right].
\end{align*}
\] (41)

For an outline of the proof within the spatial representation of the isotropic case we refer to (Miehe, 1994, A1). Again, the introduction of exclusively spherical elements in the tensor series \( A^\sharp_{1,...,n} \) or the incorporation of scalar-valued fields instead of tensorial quantities results in tangent operators which characterise an isotropic setting.

5. Cook's problem

Within the subsequent finite element example, we investigate a three-dimensional version of the classical two-dimensional Cook's membrane problem. The standard discretisation in the \( e_1, e_2 \) plane is thus expanded into the \( e_3 \) direction (when referring to a Cartesian frame). Geometry, as well as the boundary and loading conditions, are visualised in Fig. 2 whereby we choose the following parameters: \( L = 48, \ H_1 = 44, \ H_2 = 16, \ T = 4 \). The discretisation consists of \( 16 \times 16 \times 4 \) eight node bricks (Q1E9), whereby we invoke enhanced elements as advocated by Simo and Armero (1992). Furthermore, the conservative force \( F \) is considered as the resultant of a continuous shear stress with respect to the undeformed configuration. Concerning the numerical implementation, a Lagrangian parametrisation in terms of spatial fields as previously highlighted is chosen. For the sake of brevity, we make no further comments on the applied non-linear finite element setting but refer the reader to the book by Oden (1972) for more background information.

In the sequel we consider the "simple" case of a transversely isotropic material, i.e., one single normalised rank one structural tensor \( A^\sharp \) is incorporated, and we adopt an additive decomposition of the free Helmholtz energy density into a purely isotropic contribution and an anisotropic complement
\[
\psi_0^j(g^\sharp, b^\sharp, a^\sharp; X) = \psi_0^i(I_{(i),(ii),(iii)}; X) + \psi_0^j(I_{(i),...,(v)}; X),
\] (42)
see Eq. (B.1). In particular, we assume a Mooney–Rivlin term for the isotropic part
\[
\psi_0^i(I_{(i),(ii),(iii)}; X) = c_1 J_{(i)} - 3 + c_2 [J_{(ii)} - 3] + \lambda P^2 \ln(J_{(iii)}) - 2 [c_1 + c_2] \ln(\sqrt{J_{(iii)}})
\] (43)
with
\[
J_{(i)} = I_{(i)}, \quad J_{(ii)} = \frac{1}{2} [J_{(ii)}^2 - I_{(ii)}^2], \quad J_{(iii)} = \frac{1}{6} [2 I_{(ii)}^2 - 3 I_{(ii)} I_{(ii)} + I_{(ii)}^3].
\] (44)

an the additional anisotropic contribution reads
\[
\psi_0^j = \alpha \left[ J_{(iii)}^n \exp(\beta[I_{(iv)} - 1] + \delta[I_{(iv)} - 1]^2 + \varepsilon[I_{(v)} - 1] + \eta[J_{(i)} - 3][J_{(iv)} - 1]) + n[I_{(ii)} - 1] - \beta[I_{(iv)} - 1] - \varepsilon[I_{(v)} - 1] - \eta[J_{(i)} - 3][J_{(iv)} - 1]]
\] (45)
compare Almeida and Spilker (1998). The computation of appropriate stress fields and corresponding tangent operators as based on Eqs. (26), (33) is straightforward. For the following example we incorporate the material parameters
\[
c_1 = 80, \quad c_2 = 200, \quad \lambda P = 10, \quad \alpha = 1, \quad \beta = 1, \quad \delta = 0.75, \quad \varepsilon = 0.5, \quad \eta = 1, \quad n = 1.25
\] (46)
and the orientation vector \( N^\sharp : TB_0^* \rightarrow \mathbb{R} \) which defines the structural tensor via \( A^\sharp = N^\sharp \otimes N^\sharp \) reads
\[
N = 0.5883 e_1 + 0.1961 e_2 + 0.7844 e_3
\] (47)
with respect to a Cartesian frame. Since this vector, which characterises the transversely isotropic symmetry of the modelled material, does not lie in the \( e_1, e_2 \) plane, we consequently observe a severe out-of-plane deformation. Figs. 2 and 3 show different views on the deformed mesh at \( \| F \| = 4.5 \times 10^3 \). Moreover, we study the displacement of the mid point node at the top corner of the specimen, \( \| N^\sharp \| \), see Fig. 4.

In order to compare these results to an isotropic setting, we simply set \( \alpha = 0 \). Fig. 5 monitors the obtained displacement curves whereby an anisotropic setting based on standard tri-linear eight node bricks (Q1) is additionally highlighted which shows a stiffer behaviour compared to the previous computation with enhanced elements (Q1E9). Finally, the convergence of these three different settings is summarised in Table 1 whereby the residual norm of the underlying Newton iteration steps are tabulated for the load step \( \| F \| : [0, 1000] \).
Fig. 2. Cook’s problem: Anisotropic (Q1E9); geometry, boundary and loading conditions, discretisation with $16 \times 16 \times 4$ eight node bricks (left) and deformed mesh at $\|F\| = 4.5 \times 10^3$ (right).

Fig. 3. Cook’s problem: Anisotropic (Q1E9); different views on the deformed mesh at $\|F\| = 4.5 \times 10^3$.

Fig. 4. Cook’s problem: Anisotropic (Q1E9); load-displacement curve of the mid point node at the top corner.
Table 1
Cook’s problem: Residual norm for the load step $\|F\| : [0, 1000]$

<table>
<thead>
<tr>
<th>Q1E9, anisotropic</th>
<th>Q1E9, isotropic</th>
<th>Q1, anisotropic</th>
</tr>
</thead>
<tbody>
<tr>
<td>no. $|R|$</td>
<td>no. $|R|$</td>
<td>no. $|R|$</td>
</tr>
<tr>
<td>1 9.86979E+02</td>
<td>1 9.79760E+02</td>
<td>1 9.9998E+02</td>
</tr>
<tr>
<td>2 3.63984E+00</td>
<td>2 3.62822E+00</td>
<td>2 3.10966E+00</td>
</tr>
<tr>
<td>3 1.53097E+00</td>
<td>3 1.08340E−03</td>
<td>3 1.52990E−02</td>
</tr>
<tr>
<td>4 5.65937E−01</td>
<td>4 8.49209E−11</td>
<td>4 3.63277E−05</td>
</tr>
<tr>
<td>5 2.24507E−01</td>
<td>5 9.62855E−11</td>
<td></td>
</tr>
<tr>
<td>6 1.40378E−02</td>
<td></td>
<td></td>
</tr>
<tr>
<td>7 1.73148E−04</td>
<td></td>
<td></td>
</tr>
<tr>
<td>8 7.68584E−09</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Fig. 5. Cook’s problem: Load-displacement curve of the mid point node at the top corner; isotropic setting (Q1E9, left) and anisotropic setting (Q1, right).

6. Discussion

The main goal of this work is to stress on a formulation of anisotropic finite hyper-elasticity in terms of the Finger tensor and an arbitrary number of additional symmetric second order tensors (typically structural tensors). The derivation of the constitutive equation for appropriate stress fields and elastic tangent operators is based in particular on the general representation theorem of isotropic tensor functions with respect to the free Helmholtz energy density. It turns out that the stress tensors and tangent operators result in a specific additive structure in terms of the appropriate arguments of the free Helmholtz energy density. In view of numerical applications, standard spatial formulations for non-linear hyper-elasticity can thus be enlarged to anisotropic constitutive equations in a convenient manner.

Appendix A. Notation

Let $y^z_{1,...,n}$ and $z^j_{1,2}$ be second order, symmetric, contra- and co-variant tensors, respectively. The derivatives $\partial_{y^i}y^z_1$ and $\partial_{y^i}[y^z_1]^{-1}$ are then denoted by

$$\partial_{y^i}y^z_1 = g^i_j \partial_{y^j}y^z_1,$$
$$\partial_{y^i}[y^z_1]^{-1} = -g^b_{y^b}[y^z_1]^{-1} = -\frac{1}{2} \left[ [y^z_1]^{-1} \otimes [y^z_1]^{-1} + [y^z_1]^{-1} \otimes [y^z_1]^{-1} \right]$$

incorporating non-standard dyadic products of the type

$$g^i \otimes g^j = s^{ij}_1 g^i \otimes g^j, \quad g^i \otimes g^j = s^{ij}_1 g^i \otimes g^j \otimes g^j \otimes g^j,$$
$$y^z_1 \otimes y^z_1 = y^{ik}_1 y^{lj}_1 g^i \otimes g^j \otimes g^k \otimes g^l, \quad y^z_1 \otimes y^z_1 = y^{ik}_1 y^{lj}_1 g^i \otimes g^j \otimes g^k \otimes g^l.$$
In particular, the relations
\[
\mathbf{g}^a : y^a = y^a_1, \quad \mathbf{g}^b_{\{ij\}_{-1}} : y^a_2 = [y^a_1]^{-1} \cdot y^a_2 \cdot [y^a_1]^{-1}
\]
hold. The derivation for corresponding co-variant tensors \(z^b_{1,2}\) is straightforward. Furthermore, in order to keep the notation of the formulae manageable, we apply the following simplifications
\[
[y^a_{i,j,...,k} \cdot z^b_{1}]_p = [y^a_1 \cdot z^b_{1}]_p \cdot [y^a_2 \cdot z^b_{1}]_p = [[z^b_{1} \cdot y^a_{2,1,...,i}]]_p^d.
\]

As usual, transposition and symmetry operations for second and fourth order contra-variant tensors are defined as
\[
[y^a_1]_p = y^a_{1i} g_i \otimes g_j, \quad [y^a_1]_{\text{sym}} = \frac{1}{2} [y^a_1 + [y^a_1]^\dagger].
\]

\[
[y^a_1]_p = y^a_{1ij} g_i \otimes g_j \otimes g_k \otimes g_l, \quad [y^a_1]_{\text{sym}} = \frac{1}{2} [y^a + [y^a]^\dagger].
\]

Appendix B. First and second derivatives of the invariants

The following outline is based essentially on the general representation theorem of isotropic scalar-valued tensor functions, see, e.g., Boehler (1977) and references cited therein. In view of the free Helmholtz energy density as highlighted in Eq. (7), we obtain the following set of invariants \(I_{(i),(ii),(vi)}, \) whereby \(i, j \in [1, n]\) take all possible choices

(i) \( g^b : b^2 = G^2 : C^0, \)
(ii) \( [g^b : b^2 : g^b] : b^2 = G^2 : [C^0 : G^2 : C^0], \)
(iii) \( [g^b : [b^2 : g^b]^\dagger] : b^2 = G^2 : [C^0 : [G^2 : C^0]^2], \)
(iv) \( g^b : a^2_{ij} = A^2_{ij} : C^0. \)
(v) \( [g^b : b^2 : g^b] : a^2_{ij} = A^2_{ij} : C^0 : G^2 : C^0. \)
(vi) \( [g^b : a^2_{ij} : c^0] : a^2_{ij} = A^2_{ij} : C^0 : A^2_{ij} : G^2. \)

Note that the contributions (see Appendix A for notational details)

\[
[g^b : a^2_{ij} : g^b] : a^2_{ij} = A^2_{ij} : [C^0 : A^2_{ij} : C^0],
\]
\[
e^0 : a^2_{ij} = A^2_{ij} : G^0,
\]
\[
[e^0 : a^2_{ij} : c^0] : a^2_{ij} = A^2_{ij} : [G^0 : A^2_{ij} : G^0],
\]
\[
[e^0 : [a^2_{ij} : e^0] : a^2_{ij} = A^2_{ij} : [G^0 : A^2_{ij} : G^0]^2],
\]
\[
[e^0 : [a^2_{i,j,l} : e^0] : a^2_{i,j,l} = A^2_{i,j,l} : [G^0 : A^2_{i,j,l} : G^0]^3],
\]

are not represented in Eqs. (B.1) since they result in constant terms (at least for the non-dissipative case considered in this work with \( A^2_{i,...,n} = \text{const} \)). Thereby, Eqs. (B.2) incorporate once more all possible choices for \(i, j, k, l \in [1, n]\) but \(i \neq l, \) compare
Appendix A. The generators obtained by the first derivatives of the invariants with respect to the spatial metric tensor $g^0$ read as (see Appendix A for notational details)

\[
\begin{align*}
\partial_{g^0} I_{(i)} &= b^i, \\
\partial_{g^0} I_{(ii)} &= 2b^i \cdot g^0 \cdot b^i, \\
\partial_{g^0} I_{(iii)} &= 3b^i \cdot [g^0 \cdot b^i]^2, \\
\partial_{g^0} I_{(iv)} &= a_i^0, \\
\partial_{g^0} I_{(v)} &= 2[g^0 \cdot a_i^0]^\text{sym}, \\
\partial_{g^0} I_{(vi)} &= [a_{i,j}^0 \cdot c^i]_2^\text{sym}. 
\end{align*}
\]  

(B.3)

Moreover, computing the derivatives with respect to the Finger tensor $b^0$ results in

\[
\begin{align*}
\partial_{b^0} I_{(i)} &= g^0, \\
\partial_{b^0} I_{(ii)} &= 2g^0 \cdot b^0 \cdot g^0, \\
\partial_{b^0} I_{(iii)} &= 3g^0 \cdot [b^0 \cdot g^0]^2, \\
\partial_{b^0} I_{(v)} &= g^0 \cdot a^0_i \cdot g^0, \\
\partial_{b^0} I_{(vi)} &= -c^0 \cdot [a_{i,j}^0 \cdot g^0]_2^\text{sym} \cdot c^0. 
\end{align*}
\]  

(B.4)

Finally, considering the derivatives in terms of the elements of the tensor series $a_{i_1, \ldots, i_n}^0$, we end up with

\[
\begin{align*}
\partial_{a_{i_1}^0} I_{(iv)} &= g^0, \\
\partial_{a_{i}^0} I_{(v)} &= g^0 \cdot b^0 \cdot g^0, \\
\partial_{a_{j}^0} I_{(vi)} &= [g^0 \cdot a_{j}^0 \cdot c^0]_2^\text{sym}. 
\end{align*}
\]  

(B.5)

Next, the derivatives of the stress generators due to $g^0$, $b^0$ and $a_{i_1, \ldots, i_n}^0$ have to be computed which are practically second derivatives of the invariants and are again highlighted with respect to the spatial setting. In particular, one obtains (see Appendix A for notational details)

\[
\begin{align*}
\partial_{g^0}^2 \otimes \partial_{g^0} I_{(ii)} &= \left[ b^0 \otimes b^0 + b^0 \otimes b^0 \right], \\
\partial_{g^0}^2 \otimes \partial_{g^0} I_{(iii)} &= 3[b^0 \otimes [b^0 \cdot g^0] + b^0 \otimes [b^0 \cdot g^0 \cdot b^0]]^\text{SYM}, \\
\partial_{g^0}^2 \otimes \partial_{g^0} I_{(v)} &= [b^0 \otimes a_{j}^0 + b^0 \otimes a_{j}^0]^\text{SYM}. 
\end{align*}
\]  

(B.6)

for the second derivative with respect to the spatial co-variant metric tensor, compare Eqs. (B.3). Next, based on the generators in terms of the Finger tensor – Eqs. (B.4), we end up with

\[
\begin{align*}
\partial_{b^0}^2 \otimes \partial_{b^0} I_{(ii)} &= \left[ g^0 \otimes g^0 + g^0 \otimes g^0 \right], \\
\partial_{b^0}^2 \otimes \partial_{b^0} I_{(iii)} &= 3[g^0 \otimes [g^0 \cdot b^0 \cdot g^0] + g^0 \otimes [g^0 \cdot b^0 \cdot g^0]]^\text{SYM}, \\
\partial_{b^0}^2 \otimes \partial_{b^0} I_{(v)} &= \left[ c^0 \otimes [c^0 \cdot a_{j}^0 \cdot g^0] + c^0 \otimes [c^0 \cdot a_{j}^0 \cdot g^0 \cdot a_{j}^0] \right]^\text{SYM}, \\
\partial_{b^0}^2 \otimes \partial_{a_{i}} I_{(v)} &= \left[ \frac{1}{2} [g^0 \otimes g^0 + g^0 \otimes g^0] \right], \\
\partial_{b^0}^2 \otimes \partial_{a_{j}} I_{(v)} &= -\left[ \frac{1}{2} [c^0 \otimes [c^0 \cdot a_{j}^0 \cdot g^0] + c^0 \otimes [c^0 \cdot a_{j}^0 \cdot g^0]] \right]^\text{SYM}, \\
\partial_{b^0}^2 \otimes \partial_{a_{j}} I_{(v)} &= -\left[ \frac{1}{2} [c^0 \otimes [c^0 \cdot a_{j}^0 \cdot g^0] + c^0 \otimes [c^0 \cdot a_{j}^0 \cdot g^0]] \right]^\text{SYM}. 
\end{align*}
\]  

(B.7)
Finally, the derivatives with respect to the elements of the additional tensor series result in (see Eqs. (B.5))

\[
\begin{align*}
\frac{\partial^2}{\partial x^i \partial x^j} f(v) &= \frac{1}{2} [ \delta^i_j f + \delta^j_i f ] , \\
\frac{\partial^2}{\partial x^i \partial x^j} I(v) &= \frac{1}{2} [ \delta^i_j I + \delta^j_i I ] , \\
\frac{\partial^2}{\partial x^i \partial x^j} \omega (X) &= \frac{1}{2} [ \delta^i_j \omega + \delta^j_i \omega ] .
\end{align*}
\]

Appendix C. Alternative proof of the anisotropic stress relation

In the following, we reiterate an alternative proof of Proposition 4.1 for convenience of the reader. Conceptually speaking, the point of departure is based on the material covariance of the free Helmholtz energy density \( \psi^0 \) (\( C^0, G^2, A^2_{1,...,n} ; X \)). We adopt a similar outline as given in Truesdell and Noll (1965, Section 846), Šilhavý (1997, Proposition 8.2.26), Lu and Papadopoulos (2000).

We consider a material diffeomorphism \( \omega(X, t) : B_0 \times \mathbb{R} \to B_1 \) determining the linear tangent map \( F^0_\xi(X, t) : TB_0 \to TB_1 \)

with \( \det(F^0_\xi) \neq 0 \). Then, the correlated material time derivative reads \( D_t F^0_\xi = \dot{F}^0_\xi \cdot F^1_\xi \) with \( F^1_\xi : T^* B_0 \times TB_0 \to \mathbb{R} \) (see Section 3.1.1) and the common ansatz \( D_t F^0_\xi = f^0_\xi \cdot f^1_\xi = 0 \) (recall the notation \( f^2_\xi = [F^0_\xi]^{-1} \) ends up with \( D_t f^2_\xi = -f^2_\xi \cdot \dot{F}^1_\xi \).

Now, the definition of material covariance \( \psi^0_0(C^0, G^2, A^2_{1,...,n} ; X) \Rightarrow \psi^0_0(\Omega^* C^0, \omega, G^2) \Rightarrow \psi^0_0(\Omega^* C^0, \omega, G^2, A^2_{1,...,n} ; X) \) – compare Section 3 (and recall the abbreviation \( \Omega = \omega^{-1} \)) – results in the necessary condition \( D_t \psi^0_0(\Omega^* C^0, \omega, G^2, A^2_{1,...,n} ; X)|_{F^2} = D_t \psi^0_0(\omega_0 \{ \omega \}; X)|_{F^2} = 0 \forall \omega_0 \).

One obtains in particular

\[
D_t \psi^0_0(\omega_0 \{ \omega \}; X)|_{F^2} = \frac{\partial \Omega \cdot C^0}{\partial \psi^0_0} \psi^0_0 : D_t(\Omega^* C^0)|_{F^2} + \frac{\partial \omega \cdot G^2}{\partial \psi^0_0} \psi^0_0 : D_t(\omega \cdot G^2)|_{F^2} + \sum_{p=1}^{n} \frac{\partial A^2_p}{\partial \psi^0_0} \psi^0_0 : D_t(A^2_p)|_{F^2}.
\]

Next, by taking the symmetry of the terms \( \frac{\partial \omega_0 \{ \omega \} \psi^0_0}{\partial \omega_0 \{ \omega \}} \) into account, we end up with

\[
D_t \psi^0_0(\omega_0 \{ \omega \}; X)|_{F^2} = 2 \frac{\partial \Omega \cdot C^0}{\partial \psi^0_0} \psi^0_0 : [\Omega^* C^0, f^2_\xi] + \frac{\partial \omega \cdot G^2}{\partial \psi^0_0} \psi^0_0 : [\omega \cdot G^2, f^2_\xi] \quad (C.1)
\]

\[\begin{align*}
\frac{\partial}{\partial x^i} \psi^0_0(\omega_0 \{ \omega \}; X)|_{F^2} &= 2 \frac{\partial \Omega \cdot C^0}{\partial \psi^0_0} \psi^0_0 : [\Omega^* C^0, f^2_\xi] + \frac{\partial \omega \cdot G^2}{\partial \psi^0_0} \psi^0_0 : [\omega \cdot G^2, f^2_\xi] \\
&+ \sum_{p=1}^{n} \frac{\partial A^2_p}{\partial \psi^0_0} \psi^0_0 : [f^2_\xi, \omega \cdot A^2_p] \quad (C.2)
\end{align*}\]
compare Eq. (27). Hence the contribution $\Xi_{\natural}^{\sharp} \xi$ must vanish. In this context, we choose as special applications the cases $\omega = \varphi$ with $F_{\natural}^{\gamma} = F_{\sharp}$ and the identity map with $F_{\natural}^{\gamma} = G_{\sharp}$ without losing generality and obtain the anisotropic spatial and material stress relations (recall the notation $[C^{\circ}]^{-1} = B_{\sharp}$)

$$\partial_{\xi} \psi_{0}^{t} = b_{\sharp} \cdot \partial_{\xi} \psi_{0}^{t} \cdot G_{\sharp} + \sum_{p=1}^{n} a_{p}^{\sharp} \cdot \partial_{\xi} \psi_{0}^{t} \cdot G_{\sharp} = [\partial_{\xi} \psi_{0}^{t}]^{\delta}.$$

$$(C.3)$$

which obviously proves Proposition 4.1. Note that the specific choice of a regular orientation preserving material isometry with $F_{\natural}^{\gamma} \in O_{3}^{+}$ and $l_{\natural}^{\sharp} \in W^{3}$ results in the definition of a scalar-valued isotropic tensor function. Then, due to the skew-symmetry of $l_{\natural}^{\sharp}$, the contribution $\Xi_{\natural}^{\sharp} \xi$ is forced to be symmetric, i.e., $\Xi_{\natural}^{\sharp} \xi = [\Xi_{\natural}^{\sharp} \xi]^{d}$ and for an isotropic setting $\partial_{\psi}^{C_{\sharp}} \psi_{0}^{0} \cdot G^{\sharp}$ and $G^{\sharp} \cdot B_{\sharp}$ commute.

References


